[arXiv:0810.5015]
UMTG–261
LAPTH-1285/08

Analytical Bethe ansatz for the open AdS/CFT $SU(1|1)$ spin chain

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Abstract

We prove an inversion identity for the open AdS/CFT $SU(1|1)$ quantum spin chain which is exact for finite size. We use this identity, together with an analytic ansatz, to determine the eigenvalues of the transfer matrix and the corresponding Bethe ansatz equations. We also solve the closed chain by algebraic Bethe ansatz.

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1 Introduction

The factorized $SU(2|2)$-invariant bulk S-matrix [1, 2, 3] plays a central role in understanding integrability in the closed string/spin chain sector of AdS/CFT. Indeed, this S-matrix can be used to derive [2, 4, 5] the all-loop asymptotic Bethe ansatz equations (BAEs) [6] and to compute finite-size effects [7]. The corresponding factorized boundary S-matrices [8]-[13] should play a parallel role in the open string/spin chain sector. [1] The commuting open-chain transfer matrix, which is constructed from both bulk and boundary S-matrices, was recently formulated (following [20]) in [21]. Determining the eigenvalues of this transfer matrix, which has yet to be accomplished, is the key technical step to determining the corresponding all-loop BAEs.

A simpler $SU(1|1)$-invariant bulk S-matrix (which is in fact a submatrix of the $SU(2|2)$ S-matrix) was found in [6, 22], and a corresponding boundary S-matrix was found in [21]. The purpose of this paper is to determine the eigenvalues and BAEs of the open-chain transfer matrix constructed from these S-matrices [21]. We expect that this computation will serve as a useful warm-up exercise for the more realistic $SU(2|2)$ case.

An essential element of our computation is the derivation of an inversion identity for the transfer matrix $t(p)$, namely,

$$t(p)\, t(-p) = \Lambda_0(p)\, \Lambda_0(-p)\, \mathbb{I}\,, \tag{1.1}$$

where $\Lambda_0(p)$ is a known scalar function, which is an exact equation for a chain of finite size L. Similar relations (although not necessarily exact for finite size) have long been known for various lattice models [23, 24]. Together with a suitable analytic ansatz (see, e.g., [25]-[31]), the transfer matrix eigenvalues and associated BAEs can then be obtained.

The outline of this paper is as follows. In Section 2 we review the construction of the transfer matrix. In Section 3 we obtain the eigenvalues and BAEs of the transfer matrix. We also work out the weak-coupling ($g \to 0$) limit. We briefly discuss these results in Section 4. There are several appendices. In Appendix A we solve the closed chain (using algebraic Bethe ansatz), since we use the form of this solution to help formulate the analytical ansatz for the open chain. We compute the pseudovacuum eigenvalue in Appendix B. We present a proof of the inversion identity in Appendix C, and we prove in Appendix D a crossing-like identity for the transfer matrix, the dressing function being compatible with this identity.

[1] For earlier work, see [14]-[19] and references therein.

2 The transfer matrix and its properties

The basic building blocks from which the open-chain transfer matrix is constructed are bulk and boundary S-matrices. We begin this Section by reviewing these S-matrices. We then briefly review the construction of the transfer matrix and present its important properties.

2.1 Bulk S-matrix

The $SU(1|1)$ bulk S-matrix is given by [6, 22] [2]

$$S(p_1, p_2) = \begin{pmatrix} x_1^+ - x_2^- & 0 & 0 & 0 \\ 0 & x_1^- - x_2^- & (x_1^+ - x_1^-)\frac{\omega_2}{\omega_1} & 0 \\ 0 & (x_2^+ - x_2^-)\frac{\omega_1}{\omega_2} & x_1^+ - x_2^+ & 0 \\ 0 & 0 & 0 & x_1^- - x_2^+ \end{pmatrix}, \tag{2.1}$$

where $x_i^\pm = x^\pm(p_i)$, $\omega_i = \omega(p_i)$. This S-matrix is regular,

$$S(p, p) \propto \mathcal{P}\,, \tag{2.2}$$

where $\mathcal{P}$ is the graded permutation matrix,

$$\mathcal{P} = \sum_{i,j=1}^{2} (-1)^{p(i)p(j)} e_{i\,j} \otimes e_{j\,i}\,, \tag{2.3}$$

where e_{ij} is the usual elementary 2×2 matrix whose (i, j) matrix element is 1, and all others are zero; and the parity assignments are $p(1) = 0$, $p(2) = 1$. It has the unitarity property

$$S_{12}(p_1, p_2)\, S_{21}(p_2, p_1) \;=\; (x_1^+ - x_2^-)(x_2^+ - x_1^-)\, \mathbb{I} \otimes \mathbb{I} \tag{2.4}$$

where $S_{21} = \mathcal{P}_{12}\, S_{12}\, \mathcal{P}_{12}$ and $\mathbb{I}$ is the two-dimensional identity matrix; and it satisfies the graded Yang-Baxter equation (YBE) [32]

$$S_{12}(p_1, p_2)\, S_{13}(p_1, p_3)\, S_{23}(p_2, p_3) \; = S_{23}(p_2, p_3)\, S_{13}(p_1, p_3)\, S_{12}(p_1, p_2)\,, \tag{2.5}$$

where

$$S_{12} = S \otimes \mathbb{I}\,, \quad S_{13} = \mathcal{P}_{23}\, S_{12}\, \mathcal{P}_{23}\,, \quad S_{23} = \mathcal{P}_{12}\, S_{13}\, \mathcal{P}_{12}\,, \tag{2.6}$$

[2] We are not concerned here with overall scalar factors.

and $\mathcal{P}_{12} = \mathcal{P} \otimes \mathbb{I}$, $\mathcal{P}_{23} = \mathbb{I} \otimes \mathcal{P}$. The bulk S-matrix does not have, to our knowledge, true crossing symmetry. [3] However, it does obey the "crossing-like" relations

$$S_{12}^{t_1}(p_1, p_2) = \sigma_1^y \, S_{12}(p_1, p_2) \, \sigma_1^y \Big|_{x_2^+ \leftrightarrow x_2^-} , \tag{2.7}$$

$$S_{12}^{t_2}(p_1, p_2) = \sigma_2^y \, S_{12}(p_1, p_2) \, \sigma_2^y \Big|_{x_1^+ \leftrightarrow x_1^-} , \tag{2.8}$$

which imply PT symmetry

$$S_{12}^{t_1 t_2}(p_1, p_2) = -\sigma_1^y \, \sigma_2^y \, S_{21}(p_2, p_1) \, \sigma_2^y \, \sigma_1^y , \tag{2.9}$$

where t_i denotes super-transposition [33] in the i^{th} space,

$$A^t = \sum_{i,j=1}^{2} (-1)^{p(j)\big(p(i)+p(j)\big)} A_{ij} \, e_{ji} \quad \text{if} \quad A = \sum_{i,j=1}^{2} A_{ij} \, e_{ij} , \tag{2.10}$$

and the subscripts $x_\ell^+ \leftrightarrow x_\ell^-$, indicate that one has to exchange $x^+(p_\ell)$ with $x^-(p_\ell)$ in the S-matrix. Moreover, σ_i^y is the second Pauli matrix in space i.

As noted by Beisert and Staudacher [6], the YBE is satisfied without imposing any constraint between $x^+(p)$ and $x^-(p)$, and without specifying $\omega(p)$. For convenience, we henceforth set

$$\omega(p) = 1 , \tag{2.11}$$

or equivalently, we gauge $\omega(p)$ away by performing the gauge transformation [4]

$$S_{12}(p_1, p_2) \;\to\; G_1(p_1) \, G_2(p_2) \, S_{12}(p_1, p_2) \, G_2(p_2)^{-1} \, G_1(p_1)^{-1} \quad \text{with} \quad G(p) = \begin{pmatrix} \omega(p) & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.12}$$

If we regard $x^\pm(p)$ as the usual functions satisfying

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g} , \qquad \frac{x^+}{x^-} = e^{ip} , \tag{2.13}$$

[3] For the $SU(M|N)$ Yangian S-matrix $S(u) = u\mathbb{I} + i\mathcal{P}$, one can argue that crossing symmetry

$$S_{12}^{t_1}(u + \eta) = A \, S_{12}(u) \, A^{-1}$$

is possible only for $|M - N| = 2$. Indeed, canceling the u's and squaring both sides, one obtains

$$\left(\eta\mathbb{I} + i\mathcal{P}^{t_1}\right)^2 = \left(iA \, \mathcal{P} \, A^{-1}\right)^2 = -\mathbb{I} .$$

Using the fact $\left(\mathcal{P}^{t_1}\right)^2 = (M - N)\mathcal{P}^{t_1}$, it follows that $\eta = \pm i \, , M - N = \pm 2$.

[4] This transformation does not modify the eigenvalues of the transfer matrix, which we define in Sec. 2.3.

then the weak-coupling limit corresponds to setting

$$x^{\pm} = \frac{1}{g}(u \pm i/2)\,, \tag{2.14}$$

and then letting $g \to 0$. In this limit, the S-matrix (2.1) evidently reduces [6] to the well-known $SU(1|1)$ "Yangian" S-matrix,

$$S(p_1, p_2) \to \frac{1}{g} S(u_1 - u_2)\,, \qquad S(u_1 - u_2) \equiv (u_1 - u_2)\mathbb{I} + i\mathcal{P}\,, \tag{2.15}$$

where $u_i = u(p_i)$.

2.2 Boundary S-matrices

The right boundary S-matrix is given by the diagonal 2×2 matrix [21]

$$R^{-}(p) = \mathrm{diag}\left(a - x^{+}(p)\,, a + x^{-}(p)\right)\,, \tag{2.16}$$

where a is an arbitrary boundary parameter. It satisfies the (right) boundary Yang-Baxter equation [34, 35]

$$S_{12}(p_1, p_2)\, R_1^{-}(p_1)\, S_{21}(p_2, -p_1)\, R_2^{-}(p_2) = R_2^{-}(p_2)\, S_{12}(p_1, -p_2)\, R_1^{-}(p_1)\, S_{21}(-p_2, -p_1) \tag{2.17}$$

without imposing any constraint between $x^{+}(p)$ and $x^{-}(p)$ other than [8]

$$x^{\pm}(-p) = -x^{\mp}(p)\,. \tag{2.18}$$

Moreover, the left boundary S-matrix is given by [21]

$$R^{+}(p) = R^{-}(-p)\Big|_{a \mapsto b} = \mathrm{diag}\left(b + x^{-}(p)\,, b - x^{+}(p)\right)\,, \tag{2.19}$$

where b is another arbitrary boundary parameter.

In the weak-coupling limit (2.14), the boundary S-matrices reduce to

$$\begin{aligned} R^{-}(p) &\to \frac{1}{g} R^{-}(u)\,, \qquad R^{-}(u) \equiv (\alpha - i/2)\mathbb{I} - u\,\sigma^{z}\,, \\ R^{+}(p) &\to \frac{1}{g} R^{+}(u)\,, \qquad R^{+}(u) \equiv (\beta - i/2)\mathbb{I} + u\,\sigma^{z}\,, \end{aligned} \tag{2.20}$$

where we have set $a = \alpha/g$ and $b = \beta/g$.

2.3 Transfer matrix

The open-chain transfer matrix is constructed from the bulk and boundary S-matrices as follows [20, 21]

$$\begin{aligned} t(p\,;\{p_\ell\}) &= \mathrm{str}_0\, R_0^+(p)\, \mathcal{T}_0^-(p\,;\{p_\ell\}) \\ &= \mathrm{str}_0\, R_0^+(p)\, T_0(p\,;\{p_\ell\})\, R_0^-(p)\, \widehat{T}_0(p\,;\{p_\ell\})\,, \end{aligned} \tag{2.21}$$

where T and $\widehat{T}$ are a pair of monodromy matrices

$$\begin{aligned} T_0(p\,;\{p_\ell\}) &= S_{0L}(p,p_L)\cdots S_{01}(p,p_1)\,, \\ \widehat{T}_0(p\,;\{p_\ell\}) &= S_{10}(p_1,-p)\cdots S_{L0}(p_L,-p)\,, \end{aligned} \tag{2.22}$$

$\{p_1,\ldots,p_L\}$ are arbitrary "inhomogeneities" associated with each of the L quantum spaces, the auxiliary space is denoted here by 0, and str denotes supertrace [36, 37, 38]:

$$str(A) = \sum_{i=1}^{2} (-1)^{p(i)} A_{ii} = A_{11} - A_{22} \quad \text{for} \quad A = \sum_{i,j=1}^{2} A_{ij}\, e_{ij}\,. \tag{2.23}$$

The transfer matrix is constructed to have the commutativity property

$$[t(p\,;\{p_\ell\})\,, t(q\,;\{p_\ell\})] = 0 \tag{2.24}$$

for arbitrary values of p and q.

The transfer matrix also obeys the exact inversion identity

$$t(p\,;\{p_\ell\})\, t(-p\,;\{p_\ell\}) = \Lambda_0(p\,;\{p_\ell\})\, \Lambda_0(-p\,;\{p_\ell\})\, \mathbb{I}\,, \tag{2.25}$$

where $\Lambda_0(p\,;\{p_\ell\})$ is the pseudovacuum eigenvalue. This identity can easily be verified numerically for small values of L, and we prove it by recursion on L in Appendix C.

In the weak-coupling limit (2.14) with also $x_\ell^\pm = \frac{1}{g}(u_\ell \pm i/2)$, the transfer matrix (2.21) becomes [5]

$$t(u\,;\{u_\ell\}) = \mathrm{str}_0\, R_0^+(u) S_{0L}(u-u_L)\cdots S_{01}(u-u_1) R_0^-(u) S_{10}(u+u_1)\cdots S_{L0}(u+u_L)\,, \tag{2.26}$$

where the bulk and boundary S-matrices are now given by (2.15), (2.20), respectively.

[5] We suppress the overall factor $1/g^{2L+2}$.

3 Analytical Bethe ansatz

As shown in Appendix B, the pseudovacuum state consisting of all spins up

$$\Omega = \underbrace{e_1 \otimes e_1 \otimes \ldots \otimes e_1}_{L} \quad \text{where} \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1}$$

is an eigenstate of the open-chain transfer matrix (2.21), with eigenvalue

$$\begin{aligned} \Lambda_0(p\,;\{p_\ell\}) \;=\; & \frac{x^+(p)+x^-(p)}{2x^+(p)} \Big\{ (a-x^+(p))(b+x^+(p)) \prod_{\ell=1}^{L} (x^+(p)-x^-(p_\ell))(x^+(p)+x^+(p_\ell)) \\ & -(a+x^+(p))(b-x^+(p)) \prod_{\ell=1}^{L} (x^+(p)-x^+(p_\ell))(x^+(p)+x^-(p_\ell)) \Big\} \,. \end{aligned} \tag{3.2}$$

We make the "analytical ansatz" [25]-[31] that every eigenvalue of the transfer matrix can be expressed as an appropriately "dressed" pseudovacuum eigenvalue,

$$\Lambda(p\,;\{p_\ell\},\{\lambda_j\}) = \Lambda_0(p\,;\{p_\ell\})\, A(p\,;\{\lambda_j\})\,. \tag{3.3}$$

In order to determine the "dressing factor" $A(p\,;\{\lambda_j\})$, we make use of the inversion identity (2.25), which implies a corresponding identity for the eigenvalues,

$$\Lambda(p\,;\{p_\ell\},\{\lambda_j\})\, \Lambda(-p\,;\{p_\ell\},\{\lambda_j\}) = \Lambda_0(p\,;\{p_\ell\})\, \Lambda_0(-p\,;\{p_\ell\})\,. \tag{3.4}$$

It follows that the dressing factor must satisfy the constraint

$$A(p\,;\{\lambda_j\})\, A(-p\,;\{\lambda_j\}) = 1\,. \tag{3.5}$$

A natural conjecture is that the open-chain dressing factor $A(p\,;\{\lambda_j\})$ can be expressed in terms of the closed-chain dressing factor $A^{(c)}(p\,;\{\lambda_j\})$ (A.12),

$$A(p\,;\{\lambda_j\}) = \frac{A^{(c)}(p\,;\{\lambda_j\})}{A^{(c)}(-p\,;\{\lambda_j\})} = \prod_{j=1}^{M} \left(\frac{x^-(p)-x^+(\lambda_j)}{x^+(p)-x^+(\lambda_j)}\right) \left(\frac{x^-(p)+x^+(\lambda_j)}{x^+(p)+x^+(\lambda_j)}\right)\,, \tag{3.6}$$

which evidently is consistent with the constraint (3.5). As discussed in Appendix D, the dressing factor (3.6) also obeys the crossing-like relation (D.12), which provides a further consistency check. The dressing factor (3.6) obviously has poles at $p = \lambda_j$. The requirement that the eigenvalues (3.3) be analytic implies that λ_j must satisfy the open-chain BAEs

$$\begin{aligned} & \left(\frac{a-x^+(\lambda_j)}{a+x^+(\lambda_j)}\right) \left(\frac{b+x^+(\lambda_j)}{b-x^+(\lambda_j)}\right) \prod_{\ell=1}^{L} \left(\frac{x^+(\lambda_j)-x^-(p_\ell)}{x^+(\lambda_j)-x^+(p_\ell)}\right) \left(\frac{x^+(\lambda_j)+x^+(p_\ell)}{x^+(\lambda_j)+x^-(p_\ell)}\right) = 1\,, \\ & j = 1, \ldots, M\,, \quad M = 0, 1, \ldots, L\,. \end{aligned} \tag{3.7}$$

We have checked the completeness of this solution numerically for up to $L = 4$.

In the weak-coupling limit (2.14), corresponding to the transfer matrix (2.26), the solution (3.2), (3.3), (3.6), (3.7) becomes

$$\Lambda(u\,;\{u_\ell\},\{\mu_j\}) = \Lambda_0(u\,;\{u_\ell\}) \prod_{j=1}^{M} \left(\frac{u-\mu_j-i}{u-\mu_j}\right)\left(\frac{u+\mu_j}{u+\mu_j+i}\right), \tag{3.8}$$

where

$$\begin{aligned}\Lambda_0(u\,;\{u_\ell\}) \quad = \quad & \frac{u}{2(2u+i)}\Big\{(2\alpha-2u-i)(2\beta+2u+i)\prod_{\ell=1}^{L}(u-u_\ell+i)\,(u+u_\ell+i) \\ & -(2\alpha+2u+i)(2\beta-2u-i)\prod_{\ell=1}^{L}(u-u_\ell)\,(u+u_\ell)\Big\}, \end{aligned} \tag{3.9}$$

and the BAEs are given by

$$\left(\frac{2\alpha-2\mu_j-i}{2\alpha+2\mu_j+i}\right)\left(\frac{2\beta+2\mu_j+i}{2\beta-2\mu_j-i}\right)\prod_{\ell=1}^{L}\left(\frac{\mu_j-u_\ell+i}{\mu_j-u_\ell}\right)\left(\frac{\mu_j+u_\ell+i}{\mu_j+u_\ell}\right) = 1\,, \tag{3.10}$$

where we have also set $\lambda_j = \mu_j/g$.

4 Discussion

We have found an exact inversion identity for the open-chain transfer matrix constructed from the $SU(1|1)$ S-matrix [6, 22] and the corresponding boundary S-matrices [21]. We have used this inversion identity to help determine the transfer matrix eigenvalues (3.2), (3.3), (3.6) and the associated BAEs (3.7). These BAEs are evidently of the free-Fermion type, as the various Bethe roots are not coupled.

While it should also be possible to obtain these results via algebraic Bethe ansatz, we have instead pursued here the analytical Bethe ansatz approach, since the latter should be more manageable for the $SU(2|2)$ case. An inversion identity may also hold for that case. Indeed, we have verified this numerically in the weak-coupling limit with $R^{\pm}(u) = \mathbb{I}$.

Our proof of the inversion identity (for the $SU(1|1)$ case) relies on recursion on the size of the chain. It would be interesting to find a more direct proof. Unfortunately, conventional fusion techniques for graded $SU(n|m)$ chains [27, 30] do not seem to work for the $n = m$ case (see e.g., [39]) which we consider here.

Acknowledgments

This work was initiated when one of the authors (E.R.) visited the University of Miami: he wishes to thank the University and the members of the Physics Department for warm hospitality and partial support during his stay. The other author (R.N.) thanks the Galileo Galilei Institute for Theoretical Physics for hospitality and the INFN for partial support during the completion of this work. This work was supported in part by the National Science Foundation under Grants PHY-0244261 and PHY-0554821.

A The closed $SU(1|1)$ spin chain

The commuting transfer matrix for the closed $SU(1|1)$ spin chain is given by

$$t^{(c)}(p\,;\{p_\ell\}) = \mathrm{str}_0\, T_0(p\,;\{p_\ell\})\,, \tag{A.1}$$

where T is the monodromy matrix defined in (2.22), which satisfies the fundamental relation

$$S_{00'}(p,q)\, T_0(p\,;\{p_\ell\})\, T_{0'}(q\,;\{p_\ell\}) = T_{0'}(q\,;\{p_\ell\})\, T_0(p\,;\{p_\ell\})\, S_{00'}(p,q)\,. \tag{A.2}$$

The eigenvalues of the transfer matrix can easily be determined by algebraic Bethe ansatz. As usual [40], we write the monodromy matrix as a matrix in the auxiliary space

$$T_0(p\,;\{p_\ell\}) = \left(\begin{array}{cc} A(p\,;\{p_\ell\}) & B(p\,;\{p_\ell\}) \\ C(p\,;\{p_\ell\}) & D(p\,;\{p_\ell\}) \end{array}\right)\,. \tag{A.3}$$

The pseudovacuum state (3.1) consisting of all spins up is an eigenstate of both $A(p\,;\{p_\ell\})$ and $D(p\,;\{p_\ell\})$,

$$\begin{aligned} A(p\,;\{p_\ell\})\,\Omega &= \prod_{\ell=1}^{L}\left(x^+(p) - x^-(p_\ell)\right)\Omega\,, \\ D(p\,;\{p_\ell\})\,\Omega &= \prod_{\ell=1}^{L}\left(x^+(p) - x^+(p_\ell)\right)\Omega\,. \end{aligned} \tag{A.4}$$

The fundamental relation (A.2) implies the relations [6]

$$A(p\,;\{p_\ell\})B(q\,;\{p_\ell\}) = f(p,q)B(q\,;\{p_\ell\})A(p\,;\{p_\ell\}) + g(p,q)B(p\,;\{p_\ell\})A(q\,;\{p_\ell\}) \tag{A.5}$$

$$D(p\,;\{p_\ell\})B(q\,;\{p_\ell\}) = f(p,q)B(q\,;\{p_\ell\})D(p\,;\{p_\ell\}) + g(p,q)B(p\,;\{p_\ell\})D(q\,;\{p_\ell\}) \tag{A.6}$$

[6] We remark that the B operators do not commute:

$$B(p\,;\{p_\ell\})B(q\,;\{p_\ell\}) = \left(\frac{x^-(p) - x^+(q)}{x^-(q) - x^+(p)}\right) B(q\,;\{p_\ell\})B(p\,;\{p_\ell\})\,.$$

A similar non-commutativity has been observed in other graded models, see e.g. [41, 42].

where

$$f(p,q) = \frac{x^-(p) - x^+(q)}{x^+(p) - x^+(q)}\,, \qquad g(p,q) = \frac{x^+(q) - x^-(q)}{x^+(p) - x^+(q)}\,. \tag{A.7}$$

Note that the *same* functions $f(p,q)$, $g(p,q)$ appear in both (A.5) and (A.6).

Consider the state obtained by applying a set of B operators with arguments $\lambda_1, \ldots, \lambda_M$ on the pseudovacuum state,

$$|\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1\,; \{p_\ell\}) \ldots B(\lambda_M\,; \{p_\ell\})\,\Omega\,. \tag{A.8}$$

This state is an eigenstate of the transfer matrix $t^{(c)}(p\,; \{p_\ell\}) = A(p\,; \{p_\ell\}) - D(p\,; \{p_\ell\})$,

$$t^{(c)}(p\,; \{p_\ell\})\,|\lambda_1, \ldots, \lambda_M\rangle = \Lambda^{(c)}(p\,; \{p_\ell\}, \{\lambda_j\})\,|\lambda_1, \ldots, \lambda_M\rangle\,, \tag{A.9}$$

with eigenvalue

$$\Lambda^{(c)}(p\,; \{p_\ell\}, \{\lambda_j\}) = \Lambda_0^{(c)}(p\,; \{p_\ell\})\,A^{(c)}(p\,; \{\lambda_j\})\,, \tag{A.10}$$

where $\Lambda_0^{(c)}(p\,; \{p_\ell\})$ is the pseudovacuum eigenvalue

$$\Lambda_0^{(c)}(p\,; \{p_\ell\}) = \prod_{\ell=1}^{L} \left(x^+(p) - x^-(p_\ell)\right) - \prod_{\ell=1}^{L} \left(x^+(p) - x^+(p_\ell)\right)\,, \tag{A.11}$$

$A^{(c)}(p\,; \{\lambda_j\})$ is the "dressing" factor

$$A^{(c)}(p\,; \{\lambda_j\}) = \prod_{j=1}^{M} \left(\frac{x^-(p) - x^+(\lambda_j)}{x^+(p) - x^+(\lambda_j)}\right)\,, \tag{A.12}$$

and $\{\lambda_j\}$ are solutions of the closed-chain BAEs

$$\prod_{\ell=1}^{L} \left(\frac{x^+(\lambda_j) - x^-(p_\ell)}{x^+(\lambda_j) - x^+(p_\ell)}\right) = 1\,, \qquad j = 1, \ldots, M\,, \quad M = 0, 1, \ldots, L-1\,. \tag{A.13}$$

We have checked the completeness of this solution numerically for up to $L = 4$.

In the weak-coupling limit (2.14) with also $x^\pm_\ell = \frac{1}{g}(u_\ell \pm i/2)$, the closed-chain transfer matrix becomes [7]

$$t^{(c)}(u\,; \{u_\ell\}) = \mathrm{str}_0\, S_{0L}(u - u_L) \cdots S_{01}(u - u_1)\,, \tag{A.14}$$

where the S-matrix is now given by (2.15); and the solution (A.10)-(A.13) becomes

$$\begin{aligned} \Lambda(u\,; \{u_\ell\}, \{\mu_j\}) \quad &= \quad \left[\prod_{\ell=1}^{L} (u - u_\ell + i) - \prod_{\ell=1}^{L} (u - u_\ell)\right] \prod_{j=1}^{M} \left(\frac{u - \mu_j - i}{u - \mu_j}\right)\,, \\ &\prod_{\ell=1}^{L} \left(\frac{\mu_j - u_\ell + i}{\mu_j - u_\ell}\right) = 1\,, \end{aligned} \tag{A.15}$$

[7] We suppress the overall factor $1/g^L$.

where we have also set $\lambda_j = \mu_j/g$. This weak coupling limit reproduces the results obtained for periodic spin chains based on $SU(1|1)$ super-Yangian, see e.g. [41, 43, 31, 42].

We note that the closed-chain transfer matrix (A.1) does *not* satisfy an inversion identity of the form (1.1). Nevertheless, it does satisfy the relation

$$t^{(c)}(p\,;\{p_\ell\})\,\tilde{t}^{(c)}(p\,;\{p_\ell\}) = \Lambda_0^{(c)}(p\,;\{p_\ell\})\,\tilde{\Lambda}_0^{(c)}(p\,;\{p_\ell\})\,\mathbb{I}\,, \tag{A.16}$$

where the tilde ($\tilde{}$) means that one should make the replacement $x^{\pm}(p) \to x^{\mp}(p)$. The corresponding relation for the eigenvalues is evidently satisfied by the expression (A.10).

B Pseudovacuum eigenvalue

We argue here that the pseudovacuum state Ω (3.1) is an eigenstate of the open-chain transfer matrix $t(p\,;\{p_\ell\})$ (2.21), and we compute the corresponding eigenvalue $\Lambda_0(p\,;\{p_\ell\})$.

A direct calculation, usual in the context of open spin chain models, shows that $\widehat{T}_0(p\,;\{p_\ell\})\,\Omega$ is an upper triangular matrix. Then, a careful calculation of $R_0^+(p)\,\mathcal{T}_0^-(p\,;\{p_\ell\})\,\Omega$ shows, after taking the supertrace in the auxiliary space 0, that Ω is an eigenvector of the transfer matrix with eigenvalue

$$\begin{aligned}
\Lambda_0(p\,;\{p_\ell\}) \;=\;& (a-x^+)(b+x^-)\prod_{\ell=1}^{L}(x^+-x_\ell^-)(x^++x_\ell^+)\\
&-(a+x^-)(b-x^+)\prod_{\ell=1}^{L}(x^+-x_\ell^+)(x^++x_\ell^-)\\
&-\sum_{\ell=1}^{L}\Bigg\{(a-x^+)(b-x^+)(x^+-x^-)(x_\ell^+-x_\ell^-)\\
&\qquad\times\prod_{k=1}^{\ell-1}(x^+-x_k^+)(x^++x_k^-)\prod_{k=\ell+1}^{L}(x^+-x_k^-)(x^++x_k^+)\Bigg\}\,.
\end{aligned} \tag{B.1}$$

We have used the notations

$$x^{\pm} = x^{\pm}(p) \quad \text{and} \quad x_\ell^{\pm} = x^{\pm}(p_\ell)\,, \quad \ell = 1,\ldots,L\,. \tag{B.2}$$

This expression can be simplified in the following way. One first shows by recursion on L

that for any set of variables y and $z_\ell^\pm$, $\ell = 1, \dots, L$, one has the identity

$$\begin{aligned} 2y \sum_{\ell=1}^{L} (z_\ell^+ - z_\ell^-) \prod_{k=1}^{\ell-1} (y - z_k^+)(y + z_k^-) \prod_{k=\ell+1}^{L} (y - z_k^-)(y + z_k^+) \\ = \prod_{\ell=1}^{L} (y - z_\ell^-)(y + z_\ell^+) - \prod_{\ell=1}^{L} (y - z_\ell^+)(y + z_\ell^-) \,. \end{aligned} \tag{B.3}$$

Then, using this relation, one rewrites (B.1) as

$$\begin{aligned} \Lambda_0(p\,; \{p_\ell\}) &= \frac{x^+ + x^-}{2x^+} \Bigg\{ (a - x^+)(b + x^+) \prod_{\ell=1}^{L} (x^+ - x_\ell^-)(x^+ + x_\ell^+) \\ &\qquad - (a + x^+)(b - x^+) \prod_{\ell=1}^{L} (x^+ - x_\ell^+)(x^+ + x_\ell^-) \Bigg\} \,. \end{aligned} \tag{B.4}$$

C Inversion identity

We write the monodromy matrix with auxiliary space 0 and quantum spaces $1, 2, \dots, L$ as a matrix in space 0:

$$\begin{aligned} \mathcal{T}_0^{(L)}(p) &\equiv \mathcal{T}_0(p; \{p_\ell\}) = S_{0L}(p, p_L) \cdots S_{01}(p, p_1)\, R_0^-(p)\, S_{10}(p_1, -p) \cdots S_{L0}(p_L, -p) \\ &= \begin{pmatrix} A_{1\dots L}(p) & B_{1\dots L}(p) \\ C_{1\dots L}(p) & D_{1\dots L}(p) \end{pmatrix} . \end{aligned} \tag{C.1}$$

We remark that the monodromy matrix is unitary,

$$\mathcal{T}_0^{(L)}(p)\, \mathcal{T}_0^{(L)}(-p) = \rho_L(p)\, \mathbb{I}_{01\dots L} \,, \tag{C.2}$$

$$\rho_L(p) = (a - x^+)(a + x^-) \prod_{\ell=1}^{L} (x^+ + x_\ell^+)(x^+ - x_\ell^-)(x^- + x_\ell^-)(x^- - x_\ell^+) \,, \tag{C.3}$$

which in components reads

$$A_{1\dots L}(p)\, A_{1\dots L}(-p) + B_{1\dots L}(p)\, C_{1\dots L}(-p) = \rho_L(p)\, \mathbb{I}_{1\dots L} \,, \tag{C.4}$$

$$D_{1\dots L}(p)\, D_{1\dots L}(-p) + C_{1\dots L}(p)\, B_{1\dots L}(-p) = \rho_L(p)\, \mathbb{I}_{1\dots L} \,, \tag{C.5}$$

$$A_{1\dots L}(p)\, B_{1\dots L}(-p) + B_{1\dots L}(p)\, D_{1\dots L}(-p) = 0 \,, \tag{C.6}$$

$$C_{1\dots L}(p)\, A_{1\dots L}(-p) + D_{1\dots L}(p)\, C_{1\dots L}(-p) = 0 \,. \tag{C.7}$$

We decompose the scattering matrix in the same way:

$$S_{0L}(p, p_L) = \begin{pmatrix} a_L(p, p_L) & b_L(p, p_L) \\ c_L(p, p_L) & d_L(p, p_L) \end{pmatrix} \equiv \begin{pmatrix} a_L(p) & b_L(p) \\ c_L(p) & d_L(p) \end{pmatrix} , \tag{C.8}$$

where we have defined the 2×2 matrices

$$a_L(p,p_L) = \begin{pmatrix} \alpha_1(p,p_L) & 0 \\ 0 & \alpha_2(p,p_L) \end{pmatrix} \quad ; \quad b_L(p,p_L) = \begin{pmatrix} 0 & 0 \\ \beta(p,p_L) & 0 \end{pmatrix} \tag{C.9}$$

$$c_L(p,p_L) = \begin{pmatrix} 0 & \gamma(p,p_L) \\ 0 & 0 \end{pmatrix} \quad ; \quad d_L(p,p_L) = \begin{pmatrix} \delta_1(p,p_L) & 0 \\ 0 & \delta_2(p,p_L) \end{pmatrix} \tag{C.10}$$

with

$$\alpha_1(p,p_L) = x^+(p) - x^-(p_L) \ ; \ \alpha_2(p,p_L) = x^-(p) - x^-(p_L) \tag{C.11}$$

$$\beta(p,p_L) = x^+(p) - x^-(p) \ ; \ \gamma(p,p_L) = x^+(p_L) - x^-(p_L) \tag{C.12}$$

$$\delta_1(p,p_L) = x^+(p) - x^+(p_L) \ ; \ \delta_2(p,p_L) = x^-(p) - x^+(p_L) \tag{C.13}$$

or in the weak coupling limit (2.14),

$$\alpha_1(p,p_L) = \frac{u - u_L + i}{g} \ ; \ \alpha_2(p,p_L) = \frac{u - u_L}{g} \ ; \ \beta(p,p_L) = \frac{i}{g} = \gamma(p,p_L) \tag{C.14}$$

$$\delta_1(p,p_L) = \frac{u - u_L}{g} \ ; \ \delta_2(p,p_L) = \frac{u - u_L - i}{g} \tag{C.15}$$

The identity $S_{L0}(p_L,p) = S_{0L}(-p,-p_L)$ leads to the decomposition

$$S_{L0}(p_L,-p) = \begin{pmatrix} \widehat{a}_L(p) & \widehat{b}_L(p) \\ \widehat{c}_L(p) & \widehat{d}_L(p) \end{pmatrix} \tag{C.16}$$

where for any function $f(p) \equiv f(p,p_L)$, we introduced $\widehat{f}(p) \equiv f(p,-p_L)$.

The unitary relation for the S-matrix (2.4) leads to

$$a_{L+1}(p)\,\widehat{a}_{L+1}(-p) + b_{L+1}(p)\,\widehat{c}_{L+1}(-p) = -(x^+ - x^-_{L+1})(x^- - x^+_{L+1})\,\mathbb{I}_{L+1} \tag{C.17}$$

$$d_{L+1}(p)\,\widehat{d}_{L+1}(-p) + c_{L+1}(p)\,\widehat{b}_{L+1}(-p) = -(x^+ - x^-_{L+1})(x^- - x^+_{L+1})\,\mathbb{I}_{L+1} \tag{C.18}$$

$$a_{L+1}(p)\,\widehat{b}_{L+1}(-p) + b_{L+1}(p)\,\widehat{d}_{L+1}(-p) = 0 \tag{C.19}$$

$$c_{L+1}(p)\,\widehat{a}_{L+1}(-p) + d_{L+1}(p)\,\widehat{c}_{L+1}(-p) = 0 \tag{C.20}$$

Note that by changing p_{L+1} to $-p_{L+1}$, one gets a new set of relations where 'hatted' and 'unhatted' functions are exchanged. For instance, relation (C.17) leads to

$$\widehat{a}_{L+1}(p)\,a_{L+1}(-p) + \widehat{b}_{L+1}(p)\,c_{L+1}(-p) = -(x^- + x^-_{L+1})(x^+ + x^+_{L+1})\,\mathbb{I}_{L+1} \tag{C.21}$$

Then, the fundamental recursion relation

$$\mathcal{T}_0^{(L+1)}(p) = S_{0,L+1}(p,p_{L+1})\,\mathcal{T}_0^{(L)}(p)\,S_{L+1,0}(p_{L+1},-p) \tag{C.22}$$

leads to the following relations

$$
\begin{aligned}
A_{1\dots L+1}(p) &= a_{L+1}(p)\,\widehat{a}_{L+1}(p)\,A_{1\dots L}(p) - a_{L+1}(p)\,\widehat{c}_{L+1}(p)\,B_{1\dots L}(p) \\
&\quad + b_{L+1}(p)\,\widehat{a}_{L+1}(p)\,C_{1\dots L}(p) + b_{L+1}(p)\,\widehat{c}_{L+1}(p)\,D_{1\dots L}(p)\,, && \text{(C.23)} \\
B_{1\dots L+1}(p) &= a_{L+1}(p)\,\widehat{b}_{L+1}(p)\,A_{1\dots L}(p) + a_{L+1}(p)\,\widehat{d}_{L+1}(p)\,B_{1\dots L}(p) \\
&\quad + b_{L+1}(p)\,\widehat{d}_{L+1}(p)\,D_{1\dots L}(p)\,, && \text{(C.24)}
\end{aligned}
$$

$$
\begin{aligned}
C_{1\dots L+1}(p) &= c_{L+1}(p)\,\widehat{a}_{L+1}(p)\,A_{1\dots L}(p) + d_{L+1}(p)\,\widehat{a}_{L+1}(p)\,C_{1\dots L}(p) \\
&\quad + d_{L+1}(p)\,\widehat{c}_{L+1}(p)\,D_{1\dots L}(p)\,, && \text{(C.25)} \\
D_{1\dots L+1}(p) &= c_{L+1}(p)\,\widehat{b}_{L+1}(p)\,A_{1\dots L}(p) + c_{L+1}(p)\,\widehat{d}_{L+1}(p)\,B_{1\dots L}(p) \\
&\quad - d_{L+1}(p)\,\widehat{b}_{L+1}(p)\,C_{1\dots L}(p) + d_{L+1}(p)\,\widehat{d}_{L+1}(p)\,D_{1\dots L}(p)\,, && \text{(C.26)}
\end{aligned}
$$

where we have used the property (C.31) below. Let us stress that, in the r.h.s. of the above expressions, A, B, C and D act in spaces $1,\dots,L$ while a, b, c, d act in space $L+1$.

Using these expressions, it is a long but simple exercise to show by recursion on L that one has the following relations:

$$
\begin{aligned}
& A_{1\dots L}(p)\,D_{1\dots L}(-p) \sim \mathbb{I}_{1\dots L} \quad ; \quad D_{1\dots L}(p)\,A_{1\dots L}(-p) \sim \mathbb{I}_{1\dots L} && \text{(C.27)} \\
& A_{1\dots L}(p)\,A_{1\dots L}(-p) + D_{1\dots L}(p)\,D_{1\dots L}(-p) \sim \mathbb{I}_{1\dots L} && \text{(C.28)} \\
& A_{1\dots L}(p)\,C_{1\dots L}(-p) + C_{1\dots L}(p)\,D_{1\dots L}(-p) = 0 && \text{(C.29)} \\
& B_{1\dots L}(p)\,A_{1\dots L}(-p) + D_{1\dots L}(p)\,B_{1\dots L}(-p) = 0 && \text{(C.30)}
\end{aligned}
$$

where the symbol $\sim$ denotes equality up to a multiplication by a (scalar) function. The case $L=1$ can be checked directly. We show explicitly the recursion for the first equality, the other ones being proven in the same way.

We suppose that (C.27)-(C.30) are valid at a given L, and expand $A_{1\dots L+1}(p)\,D_{1\dots L+1}(-p)$ using expressions (C.23)-(C.26). From unitarity relations (C.4)-(C.7) and recursion hypothesis (C.27)-(C.30), one can eliminate terms $D_L(p)B_L(-p)$, $D_L(p)C_L(-p)$, $B_L(p)D_L(-p)$, $C_L(p)D_L(-p)$, $B_L(p)C_L(-p)$ and $C_L(p)B_L(-p)$ from any expression. We can also use the property

$$
c(p,p_1)\,U\,c(q,q_1) = 0 = b(p,p_1)\,U\,b(q,q_1)\,,\ \forall\ p,q,p_1,q_1 \quad \text{for any diagonal matrix } U, \tag{C.31}
$$

which is a generalization of the nilpotency for the matrices b and c. In the same way, since b (respectively c) are lower (respectively, upper) triangular matrices, we have

$$\begin{aligned} V\,b(p,p_1)\,U\,c(q,q_1) &= b(p,p_1)\,U\,c(q,q_1)\,V \\ V\,c(p,p_1)\,U\,b(q,q_1) &= c(p,p_1)\,U\,b(q,q_1)\,V \\ \forall\; p,q,p_1,q_1 &\qquad \text{and for any diagonal matrices } U \text{ and } V. \end{aligned} \tag{C.32}$$

As a result, one gets

$$\begin{aligned} A_{1\ldots L+1}(p)\,D_{1\ldots L+1}(-p) &= a_{L+1}(p)\,\widehat{a}_{L+1}(p)\,d_{L+1}(-p)\,\widehat{d}_{L+1}(-p)\,A_{1\ldots L}(p)\,D_{1\ldots L}(-p) \\ &-\Big(a_{L+1}(p)\,\widehat{c}_{L+1}(p)\,d_{L+1}(-p)\,\widehat{b}_{L+1}(-p)+b_{L+1}(p)\,\widehat{a}_{L+1}(p)\,c_{L+1}(-p)\,\widehat{d}_{L+1}(-p)\Big)\,\rho_L(p)\,\mathbb{I}_{1\ldots L} \\ &+b_{L+1}(p)\,\Big(\widehat{a}_{L+1}(p)\,c_{L+1}(-p)+\widehat{c}_{L+1}(p)\,d_{L+1}(-p)\Big)\,\widehat{d}_{L+1}(-p)\,D_{1\ldots L}(p)\,D_{1\ldots L}(-p) \\ &+a_{L+1}(p)\,\Big(\widehat{a}_{L+1}(p)\,c_{L+1}(-p)+\widehat{c}_{L+1}(p)\,d_{L+1}(-p)\Big)\,\widehat{b}_{L+1}(-p)\,A_{1\ldots L}(p)\,A_{1\ldots L}(-p) \\ &+a_{L+1}(p)\,\Big(\widehat{a}_{L+1}(p)\,c_{L+1}(-p)+\widehat{c}_{L+1}(p)\,d_{L+1}(-p)\Big)\,\widehat{d}_{L+1}(-p)\,A_{1\ldots L}(p)\,B_{1\ldots L}(-p) \\ &+b_{L+1}(p)\,\Big(\widehat{a}_{L+1}(p)\,c_{L+1}(-p)+\widehat{c}_{L+1}(p)\,d_{L+1}(-p)\Big)\,\widehat{b}_{L+1}(-p)\,C_{1\ldots L}(p)\,A_{1\ldots L}(-p) \\ &-\Big(b_{L+1}(p)\,\widehat{a}_{L+1}(p)\,d_{L+1}(-p)\,\widehat{d}_{L+1}(-p)+a_{L+1}(p)\,\widehat{a}_{L+1}(p)\,d_{L+1}(-p)\,\widehat{b}_{L+1}(-p)\Big)\,A_{1\ldots L}(p)\,C_{1\ldots L}(-p)\,. \end{aligned} \tag{C.33}$$

Using the explicit forms (C.11)-(C.13) (or (C.14)-(C.15) if one is interested just in the weak-coupling limit), one can check that the following relations hold:

$$\widehat{a}_{L+1}(p)\,c_{L+1}(-p)+\widehat{c}_{L+1}(p)\,d_{L+1}(-p)=0 \tag{C.34}$$

$$b_{L+1}(p)\,\widehat{a}_{L+1}(p)\,d_{L+1}(-p)\,\widehat{d}_{L+1}(-p)+a_{L+1}(p)\,\widehat{a}_{L+1}(p)\,d_{L+1}(-p)\,\widehat{b}_{L+1}(-p)=0 \tag{C.35}$$

$$a_{L+1}(p)\,\widehat{c}_{L+1}(p)\,d_{L+1}(-p)\,\widehat{b}_{L+1}(-p)+b_{L+1}(p)\,\widehat{a}_{L+1}(p)\,c_{L+1}(-p)\,\widehat{d}_{L+1}(-p)\sim\mathbb{I}_{L+1} \tag{C.36}$$

$$a_{L+1}(p)\,\widehat{a}_{L+1}(p)\,d_{L+1}(-p)\widehat{d}_{L+1}(-p)\sim\mathbb{I}_{L+1} \tag{C.37}$$

where, as above, the symbol $\sim$ denotes equality up to a multiplication by a (scalar) function. This proves that we have

$$A_{1\ldots L+1}(p)\,D_{1\ldots L+1}(-p)\;\sim\;\mathbb{I}_{1\ldots L+1}\,. \tag{C.38}$$

Let us stress that in proving relation (C.38), the identity $\mathbb{I}_{L+1}$ appearing in (C.36) and (C.37) is essential to pass from $\mathbb{I}_{1\ldots L}$ to $\mathbb{I}_{1\ldots L+1}$ in the recursion. Note also that, apart from the relation (2.18), the explicit form of $x^{\pm}(p)$ is not needed in this calculation.

The other relations are proven along the same lines, and reduce to a long list of relations on a, b, c and d that have to be fulfilled. We checked all of them. Most of these relations are ensured by the unitarity relations (C.17)-(C.20) and the following quadratic ones:

$$a_{L+1}(p)\,\widehat{a}_{L+1}(-p) - c_{L+1}(p)\,\widehat{b}_{L+1}(-p) = (x^+ - x^+_{L+1})(x^-_{L+1} - x^-)\,\mathbb{I}_{L+1} \tag{C.39}$$

$$d_{L+1}(p)\,\widehat{d}_{L+1}(-p) - b_{L+1}(p)\,\widehat{c}_{L+1}(-p) = (x^+ - x^+_{L+1})(x^-_{L+1} - x^-)\,\mathbb{I}_{L+1} \tag{C.40}$$

$$\widehat{d}_{L+1}(p)\,b_{L+1}(-p) + \widehat{b}_{L+1}(p)\,a_{L+1}(-p) = 0 \tag{C.41}$$

$$\widehat{a}_{L+1}(p)\,d_{L+1}(-p) = -(x^+ + x^+_{L+1})(x^- + x^+_{L+1})\,\mathbb{I}_{L+1} \tag{C.42}$$

For instance, once (C.42) is proved, (C.35) reduces to the unitarity relation (C.19), and (C.37) is trivially satisfied. In reducing the number of equations to be satisfied, properties (C.31) and (C.32) need also to be used.

Some quartic relations, similar to (C.36), remain to be checked directly. We verified all of them, they take two generic forms. To describe these two forms, we introduce the notation

$$g(p) \;=\; b_{L+1}(p) \text{ or } c_{L+1}(p) \tag{C.43}$$

$$\big\{h(p),\ell(p)\big\} \;=\; \big\{a_{L+1}(p),d_{L+1}(p)\big\} \text{ or } \big\{d_{L+1}(p),a_{L+1}(p)\big\} \tag{C.44}$$

Then, one can check that

$$h(p)\,\widehat{g}(p)\,a(-p)\,\widehat{h}(-p) + \ell(p)\,\widehat{d}(p)\,g(-p)\,\widehat{\ell}(-p) = 0 \tag{C.45}$$

$$h(p)\,\widehat{g}(p)\,d(-p)\,\widehat{h}(-p) + \ell(p)\,\widehat{a}(p)\,g(-p)\,\widehat{\ell}(-p) = 0 \tag{C.46}$$

together with

$$h(p)\,\widehat{h}(p)\,b(-p)\,\widehat{c}(-p) + c(p)\,\widehat{b}(p)\,\ell(-p)\,\widehat{\ell}(-p) \sim \mathbb{I} \tag{C.47}$$

$$b(p)\,\widehat{h}(p)\,h(-p)\,\widehat{c}(-p) + c(p)\,\widehat{\ell}(p)\,\ell(-p)\,\widehat{b}(-p) \sim \mathbb{I} \tag{C.48}$$

$$h(p)\,\widehat{b}(p)\,c(-p)\,\widehat{h}(-p) + \ell(p)\,\widehat{c}(p)\,b(-p)\,\widehat{\ell}(-p) \sim \mathbb{I} \tag{C.49}$$

These relations also ensure that the relations (C.27)-(C.30) are fulfilled for $L = 1$.

Once (C.27)-(C.30) are proven, it is easy to show that the transfer matrix

$$t(p;\{p_\ell\}) = \big(b + x^-(p)\big)\,A_{1\ldots L}(p) - \big(b - x^+(p)\big)\,D_{1\ldots L}(p) \tag{C.50}$$

obeys the following relation:

$$t(p;\{p_\ell\})\,t(-p;\{p_\ell\}) \sim \mathbb{I}_{12\ldots L}.$$

To determine the normalization coefficient, we apply this relation onto the pseudovacuum Ω. This leads to

$$t(p;\{p_\ell\})\,t(-p;\{p_\ell\}) = \Lambda_0(p;\{p_\ell\})\,\Lambda_0(-p;\{p_\ell\})\,\mathbb{I}_{12\ldots L}\,. \tag{C.51}$$

Finally, let us remark that this proof is also valid in the weak coupling limit, i.e. for an open spin chain based on the $SU(1|1)$ super-Yangian. Numerical investigations suggest that the inversion identity may also be valid for open spin chains based on $SU(n|n)$ super-Yangian, with trivial boundary matrices $R^{\pm}(p)\sim\mathbb{I}$.

D Crossing-like relation

We show here that the open-chain transfer matrix obeys the crossing-like relation (D.11). To this end, we use the S-matrix property (2.9) to deduce[8]

$$T_0(p)^{t_0t_1\ldots t_L} = (-1)^L\,\sigma^y_L\ldots\sigma^y_1\sigma^y_0\,\widehat{T}_0(-p)\,\sigma^y_0\sigma^y_1\ldots\sigma^y_L \tag{D.1}$$

$$\widehat{T}_0(p)^{t_0t_1\ldots t_L} = (-1)^L\,\sigma^y_L\ldots\sigma^y_1\sigma^y_0\,T_0(-p)\,\sigma^y_0\sigma^y_1\ldots\sigma^y_L \tag{D.2}$$

This implies that we have

$$t(p)^{t_1\ldots t_L} = \sigma^y_L\ldots\sigma^y_1\,str_0\Big(R^+_0(p)\,\sigma^y_0\,\widehat{T}_0(-p)\,\sigma^y_0\,R^-_0(p)\,\sigma^y_0\,T_0(-p)\,\sigma^y_0\Big)\sigma^y_1\ldots\sigma^y_L \tag{D.3}$$

Using cyclicity of the supertrace and the property

$$\sigma^y_0\,R^{\pm}_0(p)\,\sigma^y_0 = R^{\pm}_0(-p) \tag{D.4}$$

we get a first relation on the transfer matrix

$$t(p)^{t_1\ldots t_L} = \sigma^y_L\ldots\sigma^y_1\,t(-p)\,\sigma^y_1\ldots\sigma^y_L\,. \tag{D.5}$$

On the other hand, from the relation (2.8), one has

$$T_0(p)^{t_1\ldots t_L} = \sigma^y_1\ldots\sigma^y_L\,T_0(p)\Big|_{x^+\leftrightarrow x^-}\sigma^y_L\ldots\sigma^y_1 \tag{D.6}$$

$$\widehat{T}_0(p)^{t_1\ldots t_L} = \sigma^y_L\ldots\sigma^y_1\,\widehat{T}_0(p)\Big|_{x^+\leftrightarrow x^-}\sigma^y_1\ldots\sigma^y_L \tag{D.7}$$

[8]To streamline the notation, we omit the dependence on the inhomogeneity parameters.

so that one can compute

$$
\begin{aligned}
t(p)^{t_1\dots t_L} &= str_0\Big(\widehat{T}_0(p)^{t_0t_1\dots t_L}\, R_0^-(p)\, T_0(p)^{t_0t_1\dots t_L}\, R_0^+(p)\Big)\\
&= str_0\Big\{\Big(\widehat{T}_0(p)^{t_0t_1\dots t_L}\, R_0^-(p)\Big)^{t_0}\Big(T_0(p)^{t_0t_1\dots t_L}\, R_0^+(p)\Big)^{t_0}\Big\}\\
&= str_0\Big(R_0^-(p)\,\widehat{T}_0(p)^{t_1\dots t_L}\, R_0^+(p)\, T_0(p)^{t_1\dots t_L}\Big) \qquad (D.8)\\
&= \sigma_L^y\dots\sigma_1^y\; str_0\Big\{R_0^-(p)\Big(\widehat{T}_0(p)\Big)_{x^+\leftrightarrow x^-}\, R_0^+(p)\Big(T_0(p)\Big)_{x^+\leftrightarrow x^-}\Big\}\;\sigma_1^y\dots\sigma_L^y\\
&= \sigma_L^y\dots\sigma_1^y\; str_0\Big\{\Big(\widehat{T}_0(p)^{t_0}\Big)_{x^+\leftrightarrow x^-}\, R_0^-(p)\Big(T_0(p)^{t_0}\Big)_{x^+\leftrightarrow x^-}\, R_0^+(p)\Big\}\;\sigma_1^y\dots\sigma_L^y\\
&= \sigma_L^y\dots\sigma_1^y\; str_0\Big\{\sigma_0^y\Big(T_0(p)\Big)_{\substack{x^+\leftrightarrow x^-\\ x_\ell^+\leftrightarrow x_\ell^-}}\,\sigma_0^y\, R_0^-(p)\,\sigma_0^y\Big(\widehat{T}_0(p)^{t_0}\Big)_{\substack{x^+\leftrightarrow x^-\\ x_\ell^+\leftrightarrow x_\ell^-}}\,\sigma_0^y\, R_0^+(p)\Big\}\;\sigma_1^y\dots\sigma_L^y
\end{aligned}
$$

Finally, using the relations

$$
\sigma_0^y\, R_0^-(p)\,\sigma_0^y = -R_0^-(p)\Big|_{\substack{x^+\leftrightarrow x^-\\ a\to -a}} \quad\text{and}\quad \sigma_0^y\, R_0^+(p)\,\sigma_0^y = -R_0^+(p)\Big|_{\substack{x^+\leftrightarrow x^-\\ b\to -b}} \qquad (D.9)
$$

we deduce

$$
t(p)^{t_1\dots t_L} = \sigma_L^y\dots\sigma_1^y\, t(p)\,\sigma_1^y\dots\sigma_L^y\,\Big|_{\substack{x^+\leftrightarrow x^-\ ;\ a\to -a\\ x_\ell^\pm\to -x_\ell^\pm\ ;\ b\to -b}}\,. \qquad (D.10)
$$

Comparing this last equality with the relation (D.5), we arrive at the desired result

$$
t(p) = t(p)\,\Big|_{\substack{x^\pm\to -x^\pm\ ;\ a\to -a\\ x_\ell^\pm\to -x_\ell^\pm\ ;\ b\to -b}}\,. \qquad (D.11)
$$

This identity, applied to a transfer matrix eigenvector, leads to the following relation for the transfer matrix eigenvalue

$$
\Lambda(p) = \Lambda(p)\,\Big|_{\substack{x^\pm\to -x^\pm\ ;\ a\to -a\\ x_\ell^\pm\to -x_\ell^\pm\ ;\ b\to -b}\ ;\ x^+(\lambda_j)\to -x^+(\lambda_j)}\,. \qquad (D.12)
$$

Note the change of sign of the Bethe roots $x^+(\lambda_j)$ induced by the BAE (3.7). Indeed, it is easy to check that if $\{x^+(\lambda_j)\}$ is a set of solutions of these BAEs for the parameters $\{x^\pm, x_\ell^\pm, a, b\}$, then the BAE solutions for the parameters $\{-x^\pm, -x_\ell^\pm, -a, -b\}$ are given by $\{-x^+(\lambda_j)\}$.

The same calculation can be done for the closed chain. One obtains

$$
t(p) \;=\; (-1)^L\, t(p)\Big|_{\substack{x^\pm\to -x^\pm\\ x_\ell^\pm\to -x_\ell^\pm}} \quad\text{and}\quad \Lambda(p) \;=\; (-1)^L\,\Lambda(p)\Big|_{\substack{x^\pm\to -x^\pm\ ;\ x_\ell^\pm\to -x_\ell^\pm\\ x^+(\lambda_j)\to -x^+(\lambda_j)}}\,. \qquad (D.13)
$$

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